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Interpolation by Piecewise-Linear Radial Basis Functions, II

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In the two-dimensional plane, a set of nodes $x_1, x_2, ..., x_n$ is given. It is desired to interpolate arbitrary data given at the nodes by a linear combination of the functions $h_i(x) = \|x - x_i\|$. Here the norm is the l_1-norm. For this purpose, one can employ the space $\mathscr{PL}$ of all continuous piecewise-linear functions on the rectangular grid generated by the nodes. Interpolation at the nodes by this larger space is quite easy. By adding an appropriate $\mathscr{PL}$-function that vanishes on the nodes, we can obtain the linear combination of $h_1, h_2, ..., h_n$ that interpolates the data. This algorithm is much more efficient than the straightforward method of simply solving the linear system of equations $\sum c_j h_j(x_i) = d_i$. © 1991 Academic Press, Inc.

1. INTRODUCTION

Throughout the paper, $\mathscr{N}$ denotes a set of n distinct points (*nodes*) in $\mathbb{R}^2$ designated by $x_1, x_2, ..., x_n$. The basic problem of two-dimensional interpolation addressed here is as follows. A "data-function" d: $\mathscr{N} \to \mathbb{R}$ is given, and we seek a function f: $\mathbb{R}^n \to \mathbb{R}$ such that $f \mid \mathscr{N} = d$; i.e., $f(x_i) = d_i$ for $i = 1, 2, ..., n$. Such a function f is said to *interpolate d.* Usually the search for f is restricted to a class of functions that (a) are easily computed and (b) have some prescribed smoothness.

We seek an interpolant in the linear space generated by the n functions $h_j(x) = \|x - x_j\|$ $(1 \leq j \leq n)$, where the norm is chosen to be the l_1-norm.

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The existence of an interpolant $f=\sum_{j=1}^{n} c_j h_j$ for arbitrary data depends upon the invertibility of the *interpolation matrix* A, whose elements are $A_{ij}=h_j(x_i)$. In [1] it was shown that a necessary and sufficient condition for the nonsingularity of A is that $\mathcal{N}$ contain no closed rectilinear path.

Notation of [1] will be briefly reviewed here. If x is a point in $\mathbb{R}^2$, we display its coordinates by writing $x=(s, t)$. The nodes are $x_i=(s_i, t_i)$. Two coordinate projections are defined by $Px=s$ and $Qx=t$. We set

$$P(\mathcal{N})=\{\sigma_1, \sigma_2, ..., \sigma_m\}, \qquad \sigma_1<\sigma_2<\cdots<\sigma_m$$
$$Q(\mathcal{N})=\{\tau_1, \tau_2, ..., \tau_k\}, \qquad \tau_1<\tau_2<\cdots<\tau_k.$$

The rectangular grid and the rectangular hull determined by $\mathcal{N}$ are

$$RG=\{\sigma_1, \sigma_2, ..., \sigma_m\}\times\{\tau_1, \tau_2, ..., \tau_k\}$$
$$RH=\{(s, t)\colon \sigma_1\leqslant s\leqslant\sigma_m \text{ and } \tau_1\leqslant t\leqslant\tau_k\}.$$

The horizontal lines $t=\tau_j$ and the vertical lines $s=\sigma_i$ divide the plane into $(m+1)(k+1)$ rectangles, some of which are unbounded. The space $\mathcal{PL}$ consists of all continuous functions on $\mathbb{R}^2$ that are linear on each of these rectangles. The space $\mathcal{RB}$ is the linear span of the set $\{h_1, h_2, ..., h_n\}$. The functions h_i are defined by

$$h_i(x)=\|x-x_i\|=|s-s_i|+|t-t_i|.$$

A path is defined as an ordered finite set $[y_1, y_2, ..., y_q]$ in RG such that the line segments joining consecutive points have positive length and are alternately horizontal and vertical. A path is said to be closed if q is even, if $y_q\neq y_1$, and if the line segment joining y_1 with y_2 is perpendicular to the line segment joining y_1 with y_q.

2. An Equivalence Relation

In this section, we begin an analysis which leads to a description of $\mathcal{RB}$ as a subspace of $\mathcal{PL}$. The description is in the "dual" form, which is to say that $\mathcal{RB}$ will be exhibited as the intersection of hyperplanes.

Definition. An equivalence relation is introduced in $\mathcal{N}$ by declaring that two nodes are equivalent if there is a path in $\mathcal{N}$ that connects them. We also declare each node equivalent to itself.

Example. A set $\mathcal{N}$ having two equivalence classes is shown in Fig. 2.1. The following elementary lemma is given without proof.

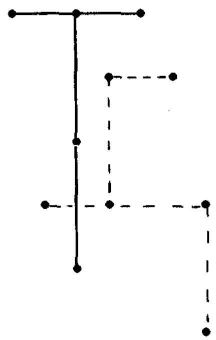

FIGURE 2.1

2.1. LEMMA. (a) *If $\mathcal{N}_0, \mathcal{N}_1, ..., \mathcal{N}_r$ are the equivalence classes that compose $\mathcal{N}$, then the sets $P(\mathcal{N}_i)$ are pairwise disjoint.*

(b) *These equivalences hold for $1 \leqslant j \leqslant n$, $0 \leqslant i \leqslant r$:*

$$s_j \in P(\mathcal{N}_i) \Leftrightarrow x_j \in \mathcal{N}_i \Leftrightarrow t_j \in Q(\mathcal{N}_i).$$

2.2. LEMMA. *If $\mathcal{N}$ contains no closed path and consists of just one equivalence class, then $m + k = n + 1$.*

Proof. Since $\mathcal{N}$ contains no closed path, we infer from 3.2 in [1] that some grid line generated by $\mathcal{N}$ contains only one node. We can assume, without loss of generality, that the horizontal line through x_1 contains no other node. We now construct a graph-theoretic tree having root x_1. At level 1 we place x_1. At level 2 we place all nodes (other than x_1) that lie on the vertical line through x_1. At level 3 we place all nodes not on levels 2 or 1 which lie on horizontal lines through the nodes on level 2. This process is continued as long as possible. For a formal description, let L_i denote the set of nodes at level i. Then $L_1 = \{x_1\}$, and recursively we put

$$L_{i+1} = [\mathcal{N} \cap Q^{-1}(Q(L_i))] \backslash [L_1 \cup \cdots \cup L_i] \qquad \text{if} \quad i \text{ is even}$$

$$L_{i+1} = [\mathcal{N} \cap P^{-1}(P(L_i))] \backslash [L_1 \cup \cdots \cup L_i] \qquad \text{if} \quad i \text{ is odd.}$$

The connections in the tree are described as follows. A node v on level $i+1$ will be joined to a node u on level i if and only if the nodes u and v lie on the same horizontal line (when i is even) or on the same vertical line (when i is odd). Every path in $\mathcal{N}$ that starts at x_1 can be traced through the successive levels of the tree. Since every node is connected to x_1 by a path, every node is in the tree.

We shall now prove that each point of L_i $(i > 1)$ accounts for one new grid line. Suppose, on the contrary, that a node y_0 in L_i does not generate a new line. Say i is odd, so that the new grid lines generated by points of L_i are vertical. Then there exists a node $z_0 \in L_j$, with $j \leqslant i$, $z_0 \neq y_0$, and $P(z_0) = P(y_0)$. By tracing backwards through the tree from z_0 to y_0 we eventually arrive at a first common node (which may be x_1). This process generates two paths

$$[z_0, z_1, ..., z_p] \quad \text{and} \quad [y_0, y_1, ..., y_q]$$

in which z_p and y_q are the same point in the tree. Since z_p is the first common node, $z_{p-1} \neq y_{q-1}$. Then

$$[z_0, z_1, ..., z_{p-1}, y_{q-1}, y_{q-2}, ..., y_0]$$

is a path. An application of 3.6 in [1] shows that this path contains a closed subpath, contrary to hypothesis.

Since each node occurs exactly once in the tree and each node generates one grid line (except for x_1 which generates two), the number of grid lines is $n+1$, the number of horizontal lines is k, and the number of vertical lines is m, and hence $n+1 = m+k$ (see Fig. 2.2). ∎

2.3. THEOREM. *Let $\mathcal{N}$ be a node set having n points and containing no closed path. If $r+1$ denotes the number of equivalence classes in $\mathcal{N}$, then* $\dim \mathcal{PL} = n + r + 4$.

Proof. By definition, $m = \# P(\mathcal{N})$ and $k = \# Q(\mathcal{N})$. By applying 2.2 to each equivalence class $\mathcal{N}_i$ we obtain

$$\# P(\mathcal{N}_i) + \# Q(\mathcal{N}_i) = \# \mathcal{N}_i + 1.$$

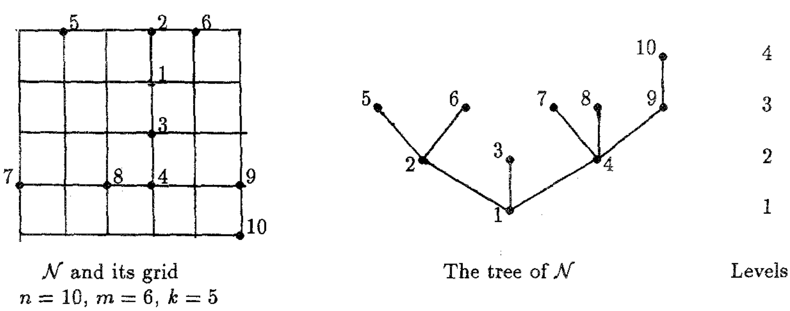

FIG. 2.2. A node set and its tree.

Since the set $P(\mathcal{N}_i)$ are pairwise disjoint and since the same is true of the sets $Q(\mathcal{N}_i)$, we have, by 2.1 in [1],

$$\dim \mathcal{PL} = m+k+3 = \sum_{i=0}^{r} [\#P(\mathcal{N}_i) + \#Q(\mathcal{N}_i)] + 3$$

$$= \sum_{i=0}^{r} [\#\mathcal{N}_i + 1] + 3 = n + r + 4. \quad \blacksquare$$

3. Annihilating Functionals for $\mathcal{RB}$

The space $\mathcal{RB}$ generated by the functions $x \mapsto \|x - x_j\|$ is a subspace of $\mathcal{PL}$. In this section we describe $\mathcal{RB}$ in the dual manner—that is, as a family of functions in $\mathcal{PL}$ that satisfy a set of homogeneous linear equations. To this end, we shall define a set of functionals $\Delta_0, ..., \Delta_{r+4}$ which annihilate $\mathcal{RB}$.

All the notation previously defined is retained here, and in addition we set

$$\lambda^{-1} = (\sigma_m - \sigma_1) + (\tau_k - \tau_1)$$

$$\sigma_0 = \sigma_1 - \lambda^{-1}, \qquad \sigma_{m+1} = \sigma_m + \lambda^{-1}$$

$$\tau_0 = \tau_1 - \lambda^{-1}, \qquad \tau_{k+1} = \tau_k + \lambda^{-1}.$$

Two definitions are given next, along with some elementary consequences without proofs.

3.1. Definition. For each $\sigma \in P(\mathcal{N})$, we define a linear functional ψ_σ which can act on any univariate piecewise linear function:

$$\psi_\sigma(u) = \lim_{s \downarrow \sigma} u'(s) - \lim_{s \uparrow \sigma} u'(s).$$

3.2. Lemma. *For the function $u(s) = |s - \alpha|$, we have $\psi_\sigma(u) = 2$ if $\alpha = \sigma$ and $\psi_\sigma(u) = 0$ if $\alpha \neq \sigma$.*

3.3. Definition. For $0 \leqslant i \leqslant r$ we define

$$\Psi_i = \sum \{\psi_\sigma : \sigma \in P(\mathcal{N}_i)\}.$$

3.4. Lemma. *Let $u(s) = |s - \alpha|$. Then $\Psi_i(u) = 2$ if $\alpha \in P(\mathcal{N}_i)$ and $\Psi_i(u) = 0$ if $\alpha \notin P(\mathcal{N}_i)$.*

In the same manner, we define functionals θ_τ and Θ_i acting on functions of t. Then we define Δ_i on $\mathscr{PL}(\mathscr{N})$ as follows. Given $f \in \mathscr{PL}$, write

$$f(s, t) = u(s) + v(t)$$

with $u \in \mathscr{PL}(P(\mathscr{N}))$ and $v \in \mathscr{PL}(Q(\mathscr{N}))$. (This expression is not unique.) Then define

$$\Delta_i(f) = \Psi_i(u) - \Theta_i(v) \qquad (0 \leqslant i \leqslant r).$$

The definition is proper, for if another expression for f is chosen it must be of the form

$$f(s, t) = [u(s) + c] + [v(s) - c]$$

for some constant c. But $\psi_\sigma(u + c) = \psi_\sigma(u)$, since ψ_σ measures the jump in the derivative at σ. Hence $\Psi_i(u + c) = \Psi_i(u)$ and similarly $\Theta_i(v - c) = \Theta_i(v)$.

3.5. LEMMA. *Each functional Δ_i, for $0 \leqslant i \leqslant r$, annihilates $\mathscr{RB}$.*

Proof. It suffices to prove that $\Delta_i(h_j) = 0$, where

$$h_j(x) = \|x - x_j\| = |s - s_j| + |t - t_j| = u(s) + v(s) \qquad (1 \leqslant j \leqslant n).$$

If $x_j \in \mathscr{N}_\alpha$ then by 2.1, s_j belongs only to $P(\mathscr{N}_\alpha)$ and t_j belongs only to $Q(\mathscr{N}_\alpha)$. Hence either $\Psi_i(u) = \Theta_i(v) = 2$ or $\Psi_i(u) = \Theta_i(v) = 0$. Consequently $\Delta_i(h_j) = 0$. ∎

We now define four additional functionals Δ_i (for $r + 1 \leqslant i \leqslant r + 4$) which annihilate $\mathscr{RB}$. Points y_i and z_i are defined as in Fig. 3.1 (which is *not* drawn to scale). The points $z_1, ..., z_4$ are at the corners of the rectangular

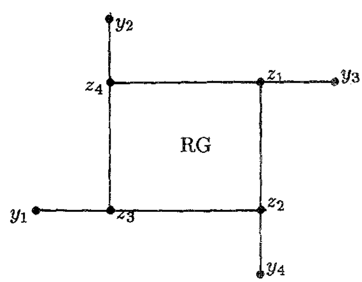

FIGURE 3.1

grid. Hence $z_1=(\sigma_m, \tau_k)$, $z_2=(\sigma_m, \tau_1)$, $z_3=(\sigma_1, \tau_1)$, and $z_4=(\sigma_1, \tau_k)$. The points $y_1, ..., y_4$ are situated as shown, and satisfy

$$\|y_1-z_3\|_1=\|y_2-z_4\|_1=\|y_3-z_1\|_1=\|y_4-z_2\|_1=\|z_2-z_4\|_1=\lambda^{-1}.$$

Now we put, for $1\leqslant i\leqslant 4$,

$$\Delta_{r+i}=\hat{y}_i+\hat{z}_i-(\hat{z}_1+\hat{z}_2+\hat{z}_3+\hat{z}_4).$$

The circumflex signifies a point-evaluation functional.

3.6. Lemma. *The four functionals $\Delta_{r+1}, ..., \Delta_{r+4}$ annihilate $\mathscr{RB}$.*

Proof. We wish to show that $\Delta_{r+i}(h_j)=0$ for $j=1, ..., n$. Fixing j, we observe that

$$\sum_{i=1}^{4} h_j(z_i)=\sum_{i=1}^{4} \|z_i-x_j\|_1=\text{perimeter of } RG.$$

For a fixed i, let z_α be the vertex opposite z_i. Then

$$\begin{aligned} h_j(y_i)+h_j(z_i) &= \|y_i-x_j\|_1+\|z_i-x_j\|_1 \\ &= \|y_i-z_\alpha\|_1+\|z_\alpha-x_j\|_1+\|x_j-z_i\|_1 \\ &= \|z_2-z_4\|_1+\|z_2-z_4\|_1 \\ &= \text{perimeter of } RG. \end{aligned}$$

By the definition of Δ_{r+i}, we have $\Delta_{r+i}(h_j)=0$. ∎

The proof that $\{\Delta_0, \Delta_1, ..., \Delta_{r+4}\}$ spans $\mathscr{RB}^\perp$ is deferred to Section 5.

4. The Subspace $\mathscr{M}$

4.1. Definitions. The subspace $\mathscr{M}$ is defined by

$$\mathscr{M}=\{f\in\mathscr{PL}(\mathscr{N}): f\mid\mathscr{N}=0\}.$$

Further definitions follow. Functions $u_0, u_1, ..., u_r$ are defined in $\mathscr{PL}(\mathbb{R})$ by specifying their knots to be $\sigma_1, ..., \sigma_m$ and specifying their values to be

$$u_i(\sigma_j)=\begin{cases}1, & \sigma_j\in\mathscr{P}(\mathscr{N}_i)\\ 0, & \text{otherwise}\end{cases} \quad (0\leqslant j\leqslant m+1).$$

A typical function u_i is graphed in Fig. 4.1. Notice that $u_i(\sigma_0)=u_i(\sigma_{m+1})=0$ for all i.

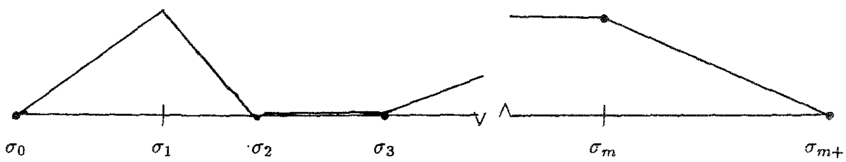

FIGURE 4.1

Functions $v_0, v_1, ..., v_r$ are defined to be piecewise linear with knots at $\tau_1, ..., \tau_k$, and having these values

$$v_i(\tau_j) = \begin{cases} 1 & \text{if } \tau_j \in Q(\mathcal{N}_i) \\ 0 & \text{otherwise} \end{cases} \quad (0 \leqslant j \leqslant k+1).$$

Then we define $g_i(s, t) = u_i(s) - v_i(t)$ for $0 \leqslant i \leqslant r$.

4.2. LEMMA. *The functions $g_0, ..., g_r$ belong to $\mathcal{M}$.*

Proof. Let x_j be any node. Let $\mathcal{N}_\alpha$ be the equivalence class containing x_j. Then $s_j \in P(\mathcal{N}_\alpha)$ and $t_j \in Q(\mathcal{N}_\alpha)$, by 2.1. We conclude that $g_i(x_j) = 0$. ∎

We now define four additional $\mathcal{PL}$ functions g_i for $r+1 \leqslant i \leqslant r+4$. Each of these vanishes on RG. At the special points y_j we assign these values:

$$g_{r+i}(y_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant 4.$$

It is clear that each of these functions belongs to $\mathcal{M}$ and the set $\{g_{r+1}, ..., g_{r+4}\}$ is linearly independent. Observe also that $\Delta_{r+i}(g_{r+j}) = \delta_{ij}$, by the definition of Δ_{r+i} in Section 3.

4.3. LEMMA. *The four functions $g_{r+1}, ..., g_{r+4}$ form a basis for the subspace of $\mathcal{PL}$ consisting of functions which vanish on the rectangular grid.*

Proof. Let f be a $\mathcal{PL}$-function such that $f \mid RG = 0$. Put $c_i = f(y_i)$. We assert that $f = \sum_1^4 c_i g_{r+i}$. For points in RG this is true since each g_{r+i} vanishes on RG. For points outside RG, we use the fact that the values of f at three vertices of a rectangle determine its value at the fourth vertex. With this, we see that f is completely determined by the four values $f(y_1), ..., f(y_4)$. For example, the values $f(y_1)$, $f(\sigma_1, \tau_1) = f(\sigma_1, \tau_2) = 0$ determine f in the strip where $\tau_1 \leqslant t \leqslant \tau_2$ and $s \leqslant \sigma_1$. ∎

4.4. LEMMA. *The function $\bar{g} = g_0 + \cdots + g_r$ vanishes on the rectangular grid.*

Proof. Observe that $\bar{g} \in \mathscr{PL}$. By 2.1 in [1], it suffices to prove that $\bar{g}$ vanishes at the points

$$(\sigma_1, \tau_j), \qquad (\sigma_\mu, \tau_1) \quad (1 \leqslant j \leqslant k, 1 \leqslant \mu \leqslant m).$$

We have

$$\bar{g}(\sigma_1, \tau_j) = \sum_{i=0}^{r} u_i(\sigma_1) - \sum_{i=0}^{r} v_i(\tau_j) = 1 - 1 = 0$$

because σ_1 belongs to exactly one set $P(\mathscr{N}_\alpha)$ and $u_\alpha(\sigma_1) = 1$, while all other $u_i(\sigma_1) = 0$. Similarly, $\sum_{i=0}^{r} v_i(\tau_j) = 1$. Analogous arguments are used at the other points. ∎

4.5. Lemma. *The set* $\{g_1, g_2, ..., g_{r+4}\}$ *is linearly independent.*

Proof. Suppose that $\sum_{i=1}^{r} c_i g_i - \sum_{i=1}^{4} a_i g_{r+i} = 0$. It follows that

$$\sum_{i=1}^{r} c_i g_i \,|\, RG = \sum_{i=1}^{4} a_i g_{r+i} \,|\, RG = 0.$$

Select a point (s, t) with $s \in P(N_j)$, $j \neq 0$, $t \in Q(\mathscr{N}_0)$. Then

$$0 = \sum_{i=1}^{r} c_i g_i(s, t) = \sum_{i=1}^{r} c_i [u_i(s) - v_i(t)] = c_j.$$

Next evaluate $\sum_{i=1}^{4} a_i g_{r+i}$ at y_j to see that $a_j = 0$. ∎

4.6. Lemma. *Let f be a $\mathscr{PL}$-function which is constant on some equivalence class, $\mathscr{N}_i$. Let $f(s, t) = u(s) + v(t)$. Then u is constant on $P(\mathscr{N}_i)$ and v is constant on $Q(\mathscr{N}_i)$.*

Proof. Let s and s' be any two points of $P(\mathscr{N}_i)$. Then there exist points t and t' such that $(s, t) \in \mathscr{N}_i$ and $(s', t') \in \mathscr{N}_i$. By the definition of an equivalence class, the points (s, t) and (s', t') are connected by an open path whose vertices lie in $\mathscr{N}_i$. On any horizontal segment of this path, say from (σ, τ) to (σ', τ), we have (since f is constant on $\mathscr{N}_i$)

$$u(\sigma) + v(\tau) = u(\sigma') + v(\tau)$$

whence $u(\sigma) = u(\sigma')$. On any vertical segment also the value of u does not change. Thus as the path is traversed, the value of u does not change, and $u(s) = u(s')$. Similarly, v is constant on $Q(\mathscr{N}_i)$. ∎

4.7. THEOREM. *Suppose that the node set $\mathcal{N}$ has exactly n poi' ., contains no closed path, and has $r+1$ equivalence classes. Then*

(i) $\mathcal{PL} = \mathcal{RB} \oplus \mathcal{M}$

(ii) $\{g_1, g_2, ..., g_{r+4}\}$ *is a basis for* $\mathcal{M}$

(iii) $\sum_{i=0}^{r+1} g_i - g_{r+2} + g_{r+3} - g_{r+4} = 0.$

Proof. Obviously $\mathcal{RB} + \mathcal{M} \subset \mathcal{PL}$. If $f \in \mathcal{PL}$ then by 7.8 in [1] there is a unique $h \in \mathcal{RB}$ that interpolates f on $\mathcal{N}$. Hence $f - h \in \mathcal{M}$ and $f \in \mathcal{RB} + \mathcal{M}$. Thus $\mathcal{PL} = \mathcal{RB} + \mathcal{M}$. That $\mathcal{RB} \cap \mathcal{M} = 0$ follows from 7.8 in [1], since the only $\mathcal{RB}$ interpolant for zero data on $\mathcal{N}$ is the 0-element of $\mathcal{RB}$.

From 2.3, $\dim \mathcal{PL} = n + r + 4$. Since $\dim \mathcal{RB} = n$, we have $\dim \mathcal{M} = r + 4$. By 4.5, $\{g_1, ..., g_{r+4}\}$ is linearly independent and therefore is a basis for $\mathcal{M}$.

Now let $\bar{g} = \sum_0^r g_i$. By 4.4, $\bar{g}$ is a $\mathcal{PL}$-function which vanishes on the rectangular grid. By 4.3, there exist coefficients α_{r+i} such that $\bar{g} = \sum_1^4 \alpha_{r+i} g_{r+i}$. By evaluating at the four points $y_1, ..., y_4$ we find that $\alpha_{r+i} = (-1)^i$. ∎

5. A DUAL ALGORITHM FOR $\mathcal{RB}$-INTERPOLATION

The direct method of computing an $\mathcal{RB}$-interpolant to a data function simply solves the interpolation equations

$$\sum_{j=1}^{n} a_j \|x_i - x_j\|_1 = d_i \qquad (1 \leqslant i \leqslant n).$$

An alternative method proceeds by first solving the interpolation problem with a function f in $\mathcal{PL}$. One can use the method of Section 4 in [1] to do this. By 4.7, f has a unique representation

$$f = h + g, \qquad h \in \mathcal{RB},\ g \in \mathcal{M}.$$

Also by 4.7, g is expressible in terms of the functions g_i, say $g = \sum_{j=1}^{r+4} a_j g_j$. The coefficients a_j can be determined from the condition $f - g \in \mathcal{RB}$, which is equivalent to

$$\Delta_i(f - g) = 0, \qquad 1 \leqslant i \leqslant r + 4$$

by 5.10 infra. These equations lead to the system

$$\sum_{j=1}^{r+4} \Delta_i(g_j) a_j = \Delta_i(f) \qquad (1 \leqslant i \leqslant r+4).$$

The invertibility of the matrix $(\Delta_i(g_j))$ follows from 7.8 in [1].

In order to carry out this dual algorithm, it will be necessary to evaluate the elements $\Delta_i(g_j)$ in the coefficient matrix. In this section these elements are computed.

Recall the definitions of Δ_i and g_j given in Sections 3 and 4. In addition, define functionals ψ_σ^+ and ψ_σ^- for any $\sigma \in \mathbb{R}$ by the equations

$$\psi_\sigma^+(u) = \lim_{s \downarrow \sigma} u'(s)$$

$$\psi_\sigma^-(u) = \lim_{s \uparrow \sigma} u'(s).$$

5.1. LEMMA. *If $0 \leqslant i \leqslant r$, then the value of $\Psi_i(u_i)$ is the sum of all terms $(\sigma_{\mu-1} - \sigma_\mu)^{-1}$ for which either*

(i) $\sigma_\mu \in P(\mathcal{N}_i)$ *and* $\sigma_{\mu-1} \notin P(\mathcal{N}_i)$ *or*

(ii) $\sigma_\mu \notin P(\mathcal{N}_i)$ *and* $\sigma_{\mu-1} \in P(\mathcal{N}_i)$.

Proof. If $\sigma_\mu \in P(\mathcal{N}_i)$ then

$$\begin{aligned}\psi_{\sigma_\mu}^+(u_i) &= [u_i(\sigma_{\mu+1}) - u_i(\sigma_\mu)](\sigma_{\mu+1} - \sigma_\mu)^{-1} \\ &= \begin{cases} (\sigma_\mu - \sigma_{\mu+1})^{-1} & \text{if } \sigma_{\mu+1} \notin P(\mathcal{N}_i) \\ 0 & \text{if } \sigma_{\mu+1} \in P(\mathcal{N}_i). \end{cases}\end{aligned}$$

Similarly,

$$\psi_{\sigma_\mu}^-(u_i) = \begin{cases} (\sigma_\mu - \sigma_{\mu-1})^{-1} & \text{if } \sigma_{\mu-1} \notin P(\mathcal{N}_i) \\ 0 & \text{if } \sigma_{\mu-1} \in P(\mathcal{N}_i). \end{cases}$$

It follows that

$$\begin{aligned}\Psi_i(u_i) &= \sum \{\psi_\sigma(u_i) : \sigma \in P(\mathcal{N}_i)\} \\ &= \sum \{\psi_\sigma^+(u_i) : \sigma \in P(\mathcal{N}_i)\} - \sum \{\psi_\sigma^-(u_i) : \sigma \in P(\mathcal{N}_i)\} \\ &= \sum \{(\sigma_\mu - \sigma_{\mu+1})^{-1} : \sigma_\mu \in P(\mathcal{N}_i) \text{ and } \sigma_{\mu+1} \notin P(\mathcal{N}_i)\} \\ &\quad - \sum \{(\sigma_\mu - \sigma_{\mu-1})^{-1} : \sigma_\mu \in P(\mathcal{N}_i) \text{ and } \sigma_{\mu-1} \notin P(\mathcal{N}_i)\} \\ &= \sum \{(\sigma_{\mu-1} - \sigma_\mu)^{-1} : \sigma_{\mu-1} \in P(\mathcal{N}_i) \text{ and } \sigma_\mu \notin P(\mathcal{N}_i)\} \\ &\quad + \sum \{(\sigma_{\mu-1} - \sigma_\mu)^{-1} : \sigma_\mu \in P(\mathcal{N}_i) \text{ and } \sigma_{\mu-1} \notin P(\mathcal{N}_i)\}. \quad \blacksquare\end{aligned}$$

5.2. LEMMA. *If* $0 \leqslant i, j \leqslant r$ *and* $i \neq j$, *then* $\Psi_i(u_j)$ *is the sum of all terms* $(\sigma_\mu - \sigma_{\mu-1})^{-1}$ *where either*

(i) $\sigma_\mu \in P(\mathcal{N}_i)$ *and* $\sigma_{\mu-1} \in P(\mathcal{N}_j)$ *or*

(ii) $\sigma_\mu \in P(\mathcal{N}_j)$ *and* $\sigma_{\mu-1} \in P(\mathcal{N}_i)$.

Proof. If $\sigma_\mu \in P(\mathcal{N}_i)$ then

$$\psi^+_{\sigma_\mu}(u_j) = u_j(\sigma_{\mu+1})(\sigma_{\mu+1} - \sigma_\mu)^{-1}.$$

This is $(\sigma_{\mu+1} - \sigma_\mu)^{-1}$ if $\sigma_{\mu+1} \in P(\mathcal{N}_j)$ and is 0 otherwise. Hence as in the preceding proof

$$\sum \{\psi^+_\sigma(u_j) \colon \sigma \in P(\mathcal{N}_i)\}$$

$$= \sum \{(\sigma_{\mu+1} - \sigma_\mu)^{-1} \colon \sigma_\mu \in P(\mathcal{N}_i) \text{ and } \sigma_{\mu+1} \in P(\mathcal{N}_j)\}$$

$$= \sum \{(\sigma_\mu - \sigma_{\mu-1})^{-1} \colon \sigma_{\mu-1} \in P(\mathcal{N}_i) \text{ and } \sigma_\mu \in P(\mathcal{N}_j)\}.$$

The calculation of $\sum \{\psi^-_\sigma(u_j) \colon \sigma \in P(\mathcal{N}_i)\}$ is similar. ∎

All of what has been proved for the matrix with elements $\Psi_i(u_j)$ can be proved for $\Theta_i(v_j)$, mutatis mutandis. Then for $0 \leqslant i, j \leqslant r$,

$$\Delta_i(g_j) = \Delta_i(u_j - v_j) = \Psi_i(u_j) - \Theta_i(-v_j) = \Psi_i(u_j) + \Theta_i(v_j).$$

In what follows $\Delta_i(g_j)$ will be computed in the remaining cases, $r < i \leqslant r+4$ and $r < j \leqslant r+4$. It is necessary to single out the equivalence classes which contain nodes on the boundary of RH. To do so, we define integers ε_ν for $0 \leqslant \nu \leqslant 4$ by the equations

$$\varepsilon_0 = \varepsilon_4, \qquad \sigma_1 \in P(\mathcal{N}_{\varepsilon_4}), \qquad \sigma_m \in P(\mathcal{N}_{\varepsilon_2}), \qquad \tau_1 \in Q(\mathcal{N}_{\varepsilon_3}), \qquad \tau_k \in Q(\mathcal{N}_{\varepsilon_1}).$$

The indices ε_1, ε_2, ε_3, ε_4 are not necessarily distinct.

5.3. LEMMA. *The following formula is valid*

$$\Delta_i(g_{r+\nu}) = \lambda(-1)^{\nu-1}\,\delta(i, \varepsilon_{\nu-1}) \qquad (0 \leqslant i \leqslant r, 1 \leqslant \nu \leqslant 4).$$

In this formula, $\lambda = \|z_2 - z_4\|^{-1}$ *and* δ *is the Kronecker symbol.*

Proof. Recall that $g_{r+\nu}$ is a $\mathcal{PL}$-function which vanishes on RG and takes these values:

$$g_{r+\nu}(y_i) = \delta_{i\nu} \qquad (1 \leqslant i, \nu \leqslant 4).$$

This function can be expressed in the form

$$g_{r+v}(s, t) = u_{r+v}(s) - v_{r+v}(t) \qquad (1 \leqslant v \leqslant 4).$$

The functions on the right in this equation take zero values on σ_j and τ_j except in these cases:

$$u_{r+1}(\sigma_0) = 1, \quad v_{r+2}(\tau_{k+1}) = -1, \quad u_{r+3}(\sigma_{m+1}) = 1, \quad v_{r+4}(\tau_0) = -1.$$

Consequently we have

$$\psi_{\sigma_1}(u_{r+1}) = (\sigma_1 - \sigma_0)^{-1} = \lambda, \quad \psi_{\sigma_m}(u_{r+3}) = \lambda,$$
$$\theta_{\tau_1}(v_{r+4}) = -\lambda, \quad \theta_{\tau_k}(v_{r+2}) = -\lambda.$$

In all other cases, $\psi_{\sigma_i}(u_{r+v}) = \theta_{\tau_j}(v_{r+v}) = 0$. Hence

$$\Psi_i(u_{r+1}) = \begin{cases} \lambda & \text{if} \quad \sigma_1 \in P(\mathcal{N}_i) \\ 0 & \text{if} \quad \sigma_1 \notin P(\mathcal{N}_i). \end{cases}$$

Likewise, $\Theta_i(v_{r+1}) = 0$, and so

$$\Delta_i(g_{r+1}) = \Psi_i(u_{r+1}) + \Theta_i(v_{r+1}) = \lambda\, \delta(i, \varepsilon_4).$$

In a similar fashion we find that

$$\Delta_i(g_{r+2}) = \Theta_i(v_{r+2}) = -\lambda\, \delta(i, \varepsilon_1)$$
$$\Delta_i(g_{r+3}) = \Psi_i(u_{r+3}) = \lambda\, \delta(i, \varepsilon_2)$$
$$\Delta_i(g_{r+4}) = \Theta_i(v_{r+4}) = -\lambda\, \delta(i, \varepsilon_3).$$

Hence the formula in the statement of the lemma is valid, with the interpretation that $\varepsilon_0 = \varepsilon_4$. ∎

5.4. LEMMA. *For $1 \leqslant v \leqslant 4$ and $0 \leqslant j \leqslant r$ we have*

$$C_{vj} = \Delta_{r+v}(g_j) = -\bar{\mu}_j + (-1)^v\, \delta(j, \varepsilon_{v-1})$$

in which $\bar{\mu}_j = \sum_{v=1}^{4} (-1)^v\, \delta(j, \varepsilon_v)$.

Proof. One can verify these formulae:

$$g_j(y_v) = (-1)^v\, \delta(j, \varepsilon_{v+2})$$
$$g_j(z_v) = (-1)^v\, \delta(j, \varepsilon_v) + (-1)^{v+1}\, \delta(j, \varepsilon_{v+1})$$
$$g_j(z_1) + g_j(z_2) + g_j(z_3) + g_j(z_4) = 2\bar{\mu}_j.$$

For example, when $\nu = 1$, the first formula is proved by observing that

$$g_j(y_1) \neq 0 \Rightarrow \tau_1 \in Q(\mathcal{N}_j) \Rightarrow j = \varepsilon_3 \Rightarrow g_j(y_1) = -v_j(\tau_1) = -1.$$

When $\nu = 3$, the second formula is established as follows. Observe that $g_j(z_3) \neq 0$ only if $j = \varepsilon_3$ or $j = \varepsilon_4$. If $j = \varepsilon_3 \neq \varepsilon_4$, then $g_j(z_3) = u_j(\sigma_1) - v_j(\tau_1) = 0 - 1 = -1$. If $j = \varepsilon_4 \neq \varepsilon_3$, then $g_j(z_3) = u_j(\sigma_1) = 1$. If $j = \varepsilon_3 = \varepsilon_4$ then $g_j(z_3) = 1 - 1 = 0$. With these formulae established, one verifies easily the assertion of the lemma. ∎

5.5. LEMMA. *If* $\sum_{i=0}^{r+4} b_i \Delta_i = 0$ *then*

$$b_{r+1} + \lambda b_{\varepsilon_4} = b_{r+2} - \lambda b_{\varepsilon_1} = b_{r+3} + \lambda b_{\varepsilon_2} = b_{r+4} - \lambda b_{\varepsilon_3} = 0.$$

Proof. We prove just one of these, viz. $b_{r+2} - \lambda b_{\varepsilon_1} = 0$. The others are similar. Construct a function v having the appearance shown in Fig. 5.1. Then put $f(s, t) = v(t)$. We have

$$\Delta_{r+\nu}(f) = \begin{cases} 1 & \text{if } \nu = 2 \\ 0 & \text{otherwise } (1 \leqslant \nu \leqslant 4) \end{cases}$$

$$\Delta_i(f) = -\Theta_i(v) = \begin{cases} -\lambda & \text{if } i = \varepsilon_1 \\ 0 & \text{otherwise } (0 \leqslant i \leqslant r). \end{cases}$$

Consequently

$$0 = \sum_{i=0}^{r+4} b_i \Delta_i(f) = -\lambda b_{\varepsilon_1} + b_{r+2}. \quad ∎$$

5.6. LEMMA. *If* $\sum_{i=0}^{r+4} b_i \Delta_i = 0$, *then*

$$b_{r+1} + b_{r+2} + b_{r+3} + b_{r+4} = (b_{\varepsilon_2} - b_{\varepsilon_4})/(\sigma_m - \sigma_1).$$

Proof. Consider the function $u \in \mathscr{PL}(\mathbb{R})$ whose graph is shown in Fig. 5.2. Put $f(s, t) = u(s)$. Then

$$f(z_3) = f(z_4) = f(y_1) = f(y_2) = 1$$
$$f(z_2) = f(z_1) = f(y_3) = f(y_4) = 0.$$

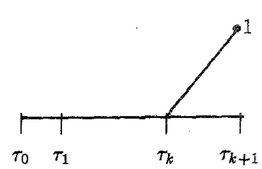

FIGURE 5.1

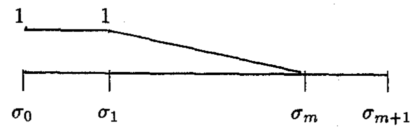

FIGURE 5.2

Consequently $\Delta_{r+\nu}(f) = -1$ for $1 \leqslant \nu \leqslant 4$. For $0 \leqslant i \leqslant r$ we have $\Delta_i(f) = 0$ in all cases with two exceptions, namely, $\Delta_{\varepsilon_2}(f) = -\Delta_{\varepsilon_4}(f) = (\sigma_m - \sigma_1)^{-1}$. Thus

$$0 = \sum_{i=0}^{r+4} b_i \Delta_i(f) = -\sum_{\nu=1}^{4} b_{r+\nu} + (b_{\varepsilon_2} - b_{\varepsilon_4})(\sigma_m - \sigma_1)^{-1}. \quad \blacksquare$$

5.7. LEMMA. *If $r+1 < n$ and if $\sum_0^{r+4} b_i \Delta_i = 0$ then $b_0 = b_1 = \cdots = b_r$.*

Proof. The hypothesis states that the number of equivalence classes is less than n. Consequently some equivalence class contains at least two elements. By a renumbering of the equivalence classes we can assume that $\#\mathcal{N}_0 \geqslant 2$. Either $\#P(\mathcal{N}_0) \geqslant 2$ or $\#Q(\mathcal{N}_0) \geqslant 2$, and we assume the former. Let a, $b \in P(\mathcal{N}_0)$, with $a < b$. Select any $j \in \{1, ..., r\}$. We shall prove that $b_j = b_0$. Select $c \in P(\mathcal{N}_j)$.

Case 1. Assume $c < a$. Define f by

$$f(s, t) = u(s) = \begin{cases} 0, & s \leqslant c \\ (a-c)^{-1}(s-c), & c < s < a \\ -(b-a)^{-1}(s-b), & a < s < b \\ 0, & s \geqslant b. \end{cases}$$

Then $f \in \mathscr{PL}$, and for each s, $f(s, \cdot)$ is constant. Furthermore, f vanishes on the eight points y_ν, z_ν. As a consequence $\Delta_{r+i}(f) = 0$ for $1 \leqslant i \leqslant 4$. For $0 \leqslant i \leqslant r$, $\Delta_i(f) = \Psi_i(u)$. Since Ψ_i measures jumps in derivatives at points of $P(\mathcal{N}_i)$, we have $\Psi_i(u) = 0$ for all i (except $i = 0$ and $i = j$) in the range $0 \leqslant i \leqslant r$. Thus after computing we have

$$0 = \sum b_i \Delta_i(f) = b_0 \Psi_0(u) + b_j \Psi_j(u) = (a-c)^{-1}(b_j - b_0).$$

Case 2. We assume that $a < c < b$. Define

$$f(s, t) = u(s) = \begin{cases} 0, & s \leqslant a \\ (c-a)^{-1}(s-a), & a \leqslant s \leqslant c \\ -(b-c)^{-1}(s-b), & c \leqslant s \leqslant b \\ 0, & s \geqslant b. \end{cases}$$

Proceeding as before we arrive at

$$0 = b_0 \Psi_0(u) + b_j \Psi_j(u) = [(b-c)^{-1} + (c-a)^{-1}](b_0 - b_j).$$

Case 3. $b < c$. The calculations are like those in Case 1. ∎

5.8. LEMMA. *Let $r+1 = n$ and $\sum_{i=0}^{r+4} b_i \Delta_i = 0$. Then for $i = 0, 1, \ldots, r$,*

$$b_i = \lambda_i b_{\varepsilon_4} + (1-\lambda_i) b_{\varepsilon_2},$$

where $\lambda_i = (\sigma_m - \sigma)/(\sigma_m - \sigma_1)$ and $\sigma \in P(\mathcal{N}_i)$.

Proof. If $i = \varepsilon_4$ or $i = \varepsilon_2$ the formula is trivial. We therefore assume $i \neq \varepsilon_4$ and $i \neq \varepsilon_2$. Then $\sigma \neq \sigma_1$ and $\sigma \neq \sigma_m$. Construct a function $u \in \mathscr{PL}(\mathbb{R})$ as shown in Fig. 5.3. Let $f(s, t) = u(s)$. Then

$$0 = \sum_{j=0}^{r+4} b_j \Delta_j(f) = b_{\varepsilon_2} \frac{1}{\sigma_m - \sigma} + b_{\varepsilon_4} \frac{1}{\sigma - \sigma_1} - b_i \left(\frac{1}{\sigma_m - \sigma} + \frac{1}{\sigma - \sigma_1} \right).$$

Consequently

$$b_{\varepsilon_2}(\sigma - \sigma_1) + b_{\varepsilon_4}(\sigma_m - \sigma) - b_i(\sigma_m - \sigma_1) = 0$$

whence

$$b_i = \lambda_i b_{\varepsilon_4} + (1-\lambda_i) b_{\varepsilon_2}. \quad ∎$$

5.9. LEMMA. *Let $r+1 = n$ and $\sum_{i=0}^{r+4} b_i \Delta_i = 0$. Then $b_0 = b_1 = \cdots = b_r$.*

Proof. From 5.7, $b_i = \lambda_i b_{\varepsilon_4} + (1-\lambda_i) b_{\varepsilon_2}$ where $0 < \lambda_i < 1$, $i = 0, \ldots, r$. Assume that $b_{\varepsilon_4} \leqslant b_{\varepsilon_2}$. From the equivalent lemma to 5.7 using the t-variable,

$$b_i = \mu_i b_{\varepsilon_1} + (1-\mu_i) b_{\varepsilon_3}, \qquad \text{where} \quad 0 < \mu_i < 1, i = 0, 1, \ldots, r.$$

From this we see that either $\varepsilon_1 = \varepsilon_4$ and $\varepsilon_2 = \varepsilon_3$ or $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = \varepsilon_4$.

Case (i). $\varepsilon_1 = \varepsilon_4$ and $\varepsilon_2 = \varepsilon_3$. Then $b_{r+1} + b_{r+2} = b_{r+3} + b_{r+4} = 0$ by 5.6, and so the previous lemma shows $b_{\varepsilon_4} = b_{\varepsilon_2}$. It then follows that $b_{\varepsilon_4} = b_{\varepsilon_2} = b_i$, $i = 0, \ldots, r$.

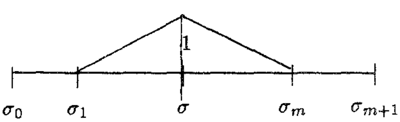

FIGURE 5.3

Case (ii). $\varepsilon_1 = \varepsilon_2$ and $\varepsilon_3 = \varepsilon_4$. Then $b_{r+2} + b_{r+3} = b_{r+1} + b_{r+4} = 0$ by 5.6, and so the previous lemma shows again that $b_{\varepsilon_4} = b_{\varepsilon_2}$. It then follows again that $b_{\varepsilon_4} = b_{\varepsilon_2} = b_i$, $i = 0, ..., r$. ▌

5.10. THEOREM. *If $\mathcal{N}$ contains no closed path, then the space $\mathcal{RB}^\perp$ is spanned by the set $\{\Delta_0, ..., \Delta_{r+4}\}$. The only dependence among these functionals, aside from a scalar multiple, is*

$$\sum_{i=0}^{r} \Delta_i + \lambda \sum_{i=1}^{4} (-1)^{i+1} \Delta_{r+i} = 0.$$

Proof. If $\sum_0^{r+4} b_i \Delta_i = 0$, then by 5.7 and 5.9, $b_0 = b_1 = \cdots = b_r$. By 5.5, $b_{r+v} = (-1)^v \lambda b_0$ for $v = 1, ..., 4$. This proves the second assertion of the theorem and that $\{\Delta_0, ..., \Delta_{r+4}\}$ spans a space of dimension $r+4$. This space is in $\mathcal{RB}^\perp$ by 3.5 and 3.6. By 2.3, we have

$$n + r + 4 = \dim \mathcal{PL} = \dim \mathcal{RB} + \dim \mathcal{RB}^\perp = n + \dim \mathcal{RB}^\perp$$

and so $\dim \mathcal{RB}^\perp = r + 4$. ▌

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